On the semigroup of monoid endomorphisms of a some extension of a bicyclic semigroup

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Let $\mathbf{B}_{\omega}^{\mathcal{F}}$ be the semigroup, which is described in [1] with the two-element family $\mathcal{F}$ of inductive subset of ω. Without loss of generality we may assume that $\mathcal{F} = \{[0,\infty),[1,\infty)\}$.

Fix an arbitrary positive integer k and any $p \in \{0,\ldots,k-1\}$. For all $i,j \in \omega$ we define the transformation $\alpha_{k,p}$ of the semigroup $\mathbf{B}_{\omega}^{\mathcal{F}}$ in the following way

$$\big(i,j,[0,\infty)\big)\alpha_{k,p} = \big(ki,kj,[0,\infty)\big) \text{ and}$$
$$\big(i,j,[1,\infty)\big)\alpha_{k,p} = \big(p+ki,p+kj,[1,\infty)\big)$$

Fix an arbitrary positive integer $k \geq 2$ and any $p \in \{1,\ldots,k-1\}\}$. For all $i,j \in \omega$ we define the transformation $\beta_{k,p}$ of the semigroup $\mathbf{B}_{\omega}^{\mathcal{F}}$ in the following way

$$\big(i,j,[0,\infty)\big)\beta_{k,p} = \big(ki,kj,[0,\infty)\big) \text{ and } \big(i,j,[1,\infty)\big)\beta_{k,p} = \big(p+ki,p+kj,[0,\infty)\big).$$

Theorem 1. ***Let*** $\boldsymbol{\mathcal{F}} = \{[\mathbf{0},\infty),[\mathbf{1},\infty)\}$ ***and*** $\boldsymbol{\varepsilon}$ ***be an injective monoid endomorphism of*** $\mathbf{B}_{\omega}^{\mathcal{F}}$***. Then either there exists a positive integer*** $\boldsymbol{k}$ ***and*** $\boldsymbol{p} \in \{\mathbf{0},\ldots,\boldsymbol{k}-\mathbf{1}\}$***, such that*** $\boldsymbol{\varepsilon} = \boldsymbol{\alpha}_{\boldsymbol{k},\boldsymbol{p}}$***, or there exist a positive integer*** $\boldsymbol{k} \geq \mathbf{2}$ ***and*** $\boldsymbol{p} \in \{\mathbf{1},\ldots,\boldsymbol{k}-\mathbf{1}\}\}$ ***such that*** $\boldsymbol{\varepsilon} = \boldsymbol{\beta}_{\boldsymbol{k},\boldsymbol{p}}$***.***

We describe the structure of the semigroup $\mathbf{End}_*^{\mathbf{1}}(\boldsymbol{B}_{\boldsymbol{\omega}}^{\boldsymbol{\mathcal{F}}})$ of injective monoid endomorphisms of the $\mathbf{B}_{\omega}^{\mathcal{F}}$**.**

Fix an arbitrary positive integer k. For all $i,j \in \omega$ we define the transformations γ_k and δ_k of the semigroup $\mathbf{B}_{\omega}^{\mathcal{F}}$ in the following way

$$\big(i,j,[0,\infty)\big)\gamma_{k,p} = \big(i,j,[1,\infty)\big)\gamma_{k,p} = \big(ki,kj,[0,\infty)\big);$$
$$\big(i,j,[0,\infty)\big)\delta_{k,p} = \big(ki,kj,[0,\infty)\big) \text{ and}$$
$$\big(i,j,[1,\infty)\big)\delta_{k,p} = \big(k(i+1),k(j+1),[0,\infty)\big)$$

Theorem 2. ***If*** $\boldsymbol{\mathcal{F}} = \{[\mathbf{0},\infty),[\mathbf{1},\infty)\}$***, then for any non-injective monoid endomorphism*** $\boldsymbol{\varepsilon}$ ***of the monoid*** $\mathbf{B}_{\omega}^{\mathcal{F}}$ ***only one of the following conditions holds:***

(1) $\boldsymbol{\varepsilon}$ ***is the annihilating endomorphism, i.e.,*** $\boldsymbol{\varepsilon} = \boldsymbol{\gamma}_{\mathbf{0}} = \boldsymbol{\delta}_{\mathbf{0}}$***;***

(2) $\boldsymbol{\varepsilon} = \boldsymbol{\gamma}_{\boldsymbol{k}}$ ***for some positive integer*** $\boldsymbol{k}$***;***

(3) $\boldsymbol{\varepsilon} = \boldsymbol{\delta}_{\boldsymbol{k}}$ ***for some positive integer*** $\boldsymbol{k}$***.***

We describe the structure of the semigroup $\mathbf{End}_*(\boldsymbol{B}_{\boldsymbol{\omega}}^{\boldsymbol{\mathcal{F}}})$ of non-injective monoid endomorphisms of $\mathbf{B}_{\omega}^{\mathcal{F}}$**.**

Also, we describe the structure of the semigroup $\mathbf{End}(\boldsymbol{B}_{\boldsymbol{\omega}}^{\boldsymbol{\mathcal{F}}})$ of all monoid endomorphisms of $\mathbf{B}_{\omega}^{\mathcal{F}}$.

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Lagrangian approach for Navier-Stokes Equations

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Smoothed Particle Hydrodynamics (SPH) is a Lagrangian method gaining popularity in fields ranging from entertainment to engineering. It perfectly handles complex scenarios like free-surface fluids with dynamic boundaries and is useful in both special effects and engineering. This makes SPH ideal for solving the Navier-Stokes equations in this study.

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f}_{\text{ext}} \quad (1)$$

where ρ - dynamic viscosity, $\mathbf{v}$ - velocity, p - pressure, μ - viscosity, $\mathbf{f}_{\text{ext}}$ - external forces.

The concept of Smoothed Particle Hydrodynamics (SPH) can be generally described as a method for discretizing spatial field quantities and spatial differential operators, such as gradients, divergence and curl.

Dirac-δ identity is the basis for the discretization. For continuous compactly supported function $A(\mathbf{x})$:

$$A(\mathbf{x}) = (A * \delta)(\mathbf{x}) = \int A(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')\, dv' \quad (2)$$

where dv' denotes the volume integration variable corresponding to $\mathbf{x}'$.

To address the challenges of discretizing the Dirac-δ function, which is neither a conventional function nor can be discretized, we first approximate $\delta(r)$ continuously using a kernel function $W(r, h)$, where h denotes the kernel's smoothing length as proposed in [1]. Such that:

$$\lim_{h \to 0} W(r, h) = \delta(r) \quad (3)$$

Final step for the SPH discretization involves substituting of the analytical integral in Eq. (2) with a sum over discrete sampling points as follows:

$$(A * W)(\mathbf{x}_i) = \int \frac{A(\mathbf{x}')}{\rho(\mathbf{x}')} W(\mathbf{x} - \mathbf{x}', h)\rho(\mathbf{x}')\, d\mathbf{v}' \approx \sum_j A_j \frac{m_j}{\rho_j} W(\mathbf{x}_i - \mathbf{x}_j, h) \quad (4)$$

Laplace operator can be discretized using Eq. (4) as:

$$\nabla^2 A_i \approx \sum_j \frac{m_j}{\rho_j} A_j \nabla^2 W_{ij} \quad (5)$$

However, this approach results in a relatively poor estimate of the second-order differential. An improved discrete operator for the Laplacian was introduced by Brookshaw in [2]. The core concept behind this formulation is to utilize only the first-order derivative of the kernel function and approximate the second derivative through a finite-difference-like operation, specifically by dividing by the particle distance:

$$\nabla^2 A_i \approx -\sum_j \frac{m_j}{\rho_j} A_{ij} \frac{2\|\nabla W_{ij}\|}{\|r_{ij}\|} \tag{6}$$

We can now create a basic simulator for weakly compressible fluids using SPH and symplectic Euler integration:

- for all *particle i*
  Reconstruct density ρ_i at $\mathbf{x}_i$ using Eq. (4)
- for all *particle i*
  Compute $F_i^{\text{viscosity}} = m_i \frac{\mu}{\rho_i} \nabla^2 \mathbf{v}_i$ using Eq. (6)
  Assign $\mathbf{v}_i^* = \mathbf{v}_i + \frac{\Delta t}{m_i}(F_i^{\text{viscosity}} + F_i^{\text{ext}})$
- for all *particle i*
  Compute $F_i^{\text{pressure}} = -\frac{1}{\rho}\nabla p$ using state equation
- for all *particle i*
  $\mathbf{v}_i(t+\Delta t) = \mathbf{v}_i^* + \frac{\Delta t}{m_i} F_i^{\text{pressure}}$
  $\mathbf{x}_i(t+\Delta t) = \mathbf{x}_i + \Delta t \mathbf{v}_i(t+\Delta t)$

In conclusion, this paper provided an introduction to the Navier-Stokes equations and Smoothed Particle Hydrodynamics (SPH), outlining how to discretize the Laplace operator and illustrating a basic algorithm for fluid simulation.

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2. Brookshaw L., A method of calculating radiative heat diffusion in particle simulations. Publications of the Astronomical Society of Australia. - 1985. - 6, 2. - p. 207–210.